Directed lines in sparse potentials

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(Received 20 May 1996)

We present a continuum formulation of a $(d+1)$-dimensional directed line interacting with sparse potentials (i.e., d-dimensional potentials defined only at discrete longitudinal locations.) An iterative solution for the partition function is derived. The impulsive influence of the potentials induces discontinuities in the evolution of the probability density $P(\mathbf{x},t)$ of the directed line. The effects of these discontinuities are studied in detail for the simple case of a single defect. We then investigate sparse columnar potentials defined as a periodic array of defects in $(2+1)$ dimensions, and solve exactly for P. A nontrivial binding-unbinding transition is found. [S1063-651X(97)04101-9]

PACS number(s): 05.40.+j, 03.65.−w, 68.35.Rh, 36.20.−r

I. INTRODUCTION

The physics of directed lines (or directed polymers) has been the focus of much interest over the past two decades. This is mainly due to the very wide range of applicability of these simple models to important physical processes, such as wetting [1], the motion of domain walls in magnets [2], and the physics of flux lines in superconductors [3]. The interest has been intensified by attempting to understand the effect of disordered potentials in such systems. In superconductors, for instance, it is well known that point disorder can help to localize the flux lines, hence allowing true superconducting current flow; and recently, it has been realized that columnar defects provide an even better mechanism for pinning [4]. At the model level, the addition of disorder allows nonobvious connections to be made to other physical systems such as spin glasses [5], nonequilibrium interface growth [6], and shock waves in fluids (in terms of the noisy Burgers equation [7]). A quantitative understanding of the effect of such types of disorder on directed lines is still lacking.

Aside from these important applications, the model of a single directed line interacting with an external potential is of substantial interest in its own right. There has been a great deal of work on the purely theoretical front in trying to understand these systems, with approaches ranging from lattice RSOS (restricted solid-on-solid) descriptions [1], to powerful renormalization group studies [8], and phenomenological scaling arguments [9]. As indicated above, what is still lacking is a systematic way of treating strongly disordered potentials; although much progress may be made in $(1+1)$ dimensions [1,10]. The overwhelming difficulties present in the analytic study of directed polymers may be countered to some degree by simplifying some aspects of the problem without trivializing the physics. One possibility is to simplify the form of the external potential. For instance, the problem of bulk disorder is of enormous interest, but completely intractable above $(1+1)$ dimensions. One may then consider simpler scenarios (such as a very dilute limit of point defects, or columnar disorder) where controlled analytic calculations may provide precious insight. In fact, such is the richness of the physics of directed lines, that even for the simple example of a nondisordered columnar potential, there exists nontrivial behavior of the line, especially as a function of spatial dimension [8,11,12].

In this paper we shall introduce a certain class of models that is best described as ''a directed line interacting with sparse potentials.'' To clarify this, consider a directed line in $(d+1)$ dimensions. We label the transverse directions by a position vector $\mathbf{y}$ and the longitudinal direction by a scalar s. The line generally exists in the presence of a potential $V(\mathbf{y},s)$. Our use of the term ''sparse potentials'' corresponds to the following form for V:

$$V(\mathbf{y},s)=\sum_{n=1}^{\infty} v_n(\mathbf{y})\,\delta(s-\tau_n),$$

where at this level of discussion the positions $\{\tau_n\}$ of the ''impulses'' v_n, along with the functional form of the impulses themselves are left free. Specific choices for these quantities may be made which then correspond to physically realizable systems. For instance, taking the $\{\tau_n\}$ to be regularly spaced leads one to consider a directed line interacting with a set of layer potentials, as might be found in a regular crystal. Alternatively, one may take the $\{\tau_n\}$ to be drawn from some distribution function, which along with taking $v_n(\mathbf{y})\sim\delta(\mathbf{y}-\mathbf{y_n})$, corresponds to a directed line interacting with a very dilute set of point defects, as may be realized in an ''almost-pure'' superconducting single crystal.

Towards the end of the paper, we shall be interested in a specific subclass of sparse potentials, namely, periodic columnar arrays of point defects, where exact solutions are possible in the physically relevant case of $(2+1)$ dimensions. Before such specialization, we concern ourselves with a more general analysis of the continuum theory of sparse potentials. An important point is that the sparse potentials have an impulsive action on the probability density of the directed line, leading to discontinuities in this function. Along with this effect, is the physical constraint that the discontinuity in the density must never be such as to make the density negative. The short-scale consistency of the theory at these impulses will be seen to have dramatic effects on the global properties of the probability density of the line.

The outline of the remainder of the paper is as follows. In the next section, we provide a general continuum formula-

tion of the problem following the standard methods of a path-integral description. Section III is concerned with constructing an iterative solution for the partition function for a given set of sparse potentials, due account being taken of the discontinuous nature of this function. In Sec. IV, we concentrate on the simple case of a single sparse potential that is taken to be very short ranged in the transverse directions (i.e., approaching a δ function.) This corresponds to a single point defect, and is exactly solvable. Simple as it may be, this case reveals the subtlety of the discontinuous nature of the probability density and allows us some intuition into the qualitative difference of effects between attractive defects (whose effect may be global) and repulsive defects (whose effect is always local.) In Secs. V and VI we concentrate on the case of a periodic array of potentials, each of which is taken to be short ranged — this is essentially a columnar array of point defects. Section V sets out the solution to the case of all the defects being either attractive or repulsive, while Sec. VI is concerned with the solution to the case of alternating positive and negative defects. We restrict our attention to the physically relevant case of $(2+1)$ dimensions, which by good fortune is the most analytically tractable. Comparison is made between our results for these models and recent investigations of related microscopic models [13,14]. We end the paper with our conclusions and a discussion of extensions to the present work.

II. FORMULATION OF THE MODEL

As mentioned in the Introduction, there are a number of model descriptions of directed lines, with the main difference being whether one chooses to work in the continuum or on a lattice. In this paper we shall use a continuum formulation, which has the advantage that derived results will not be dependent upon microscopic parameters, and thus may be hoped to have some universal applicability.

We consider a directed line in a $(d+1)$-dimensional space, with $\mathbf{y}$ labeling the d transverse directions, and s labeling the longitudinal direction. We demand that the line begins at the point $(\mathbf{0},0)$ and ends at $(\mathbf{x},t)$. The restricted partition function for the line, in the presence of a potential $V(\mathbf{y},s)$ is given by

$$Z(\mathbf{x},t) = \int_{\mathbf{y}(0)=\mathbf{0}}^{\mathbf{y}(t)=\mathbf{x}} \mathcal{D}\mathbf{y} \exp\left\{ -\frac{1}{T}\int_0^t ds \left[\kappa\left(\frac{d\mathbf{y}}{ds}\right)^2 + V(\mathbf{y}(s),s)\right]\right\}, \tag{1}$$

where T represents temperature, and κ is the elastic constant for the line.

Following standard methods [15] we may rewrite this path integral in the form of a partial differential equation (PDE). Explicitly one finds

$$\partial_t Z(\mathbf{x},t) = \nu\nabla^2 Z - V(\mathbf{x},t)Z, \tag{2}$$

with boundary condition $Z(\mathbf{x},0)=\delta^d(\mathbf{x})$. We have defined a diffusion constant $\nu = T/4\kappa$, and have scaled V so as to absorb a factor of $1/T$.

At this point we specialize to the system of interest, i.e., the case where V represents a set of sparse potentials. Explicitly we write

$$V(\mathbf{y},s) = \sum_{n=1}^{\infty} v_n(\mathbf{y})\,\delta(s-\tau_n), \tag{3}$$

where $\{\tau_n\}$ represent the longitudinal locations of the potentials $\{v_n(\mathbf{y})\}$. As a final step we integrate Eq. (2) using the Green function of the diffusion equation $g(\mathbf{x},t) = (4\pi\nu t)^{-d/2}\exp(-x^2/4\nu t)$, which yields an integral equation for Z of the form

$$Z(\mathbf{x},t) = g(\mathbf{x},t) - \int d^d x' \int_0^t dt' g(\mathbf{x}-\mathbf{x}',t-t') \times \sum_{n=1}^{\infty} v_n(\mathbf{x}')\,\delta(t'-\tau_n) Z(\mathbf{x}',t'). \tag{4}$$

It appears to be a simple matter to integrate over the variable t' using the δ functions, but as we shall see in the next section, this must be done with some care.

We end this section with a discussion of the physical quantities that one can obtain from Z. It is important to realize that the restricted partition function itself is not a physical quantity. In order to be meaningful, it must be normalized. We therefore construct the probability density of the directed lines via

$$P(\mathbf{x},t) = \frac{Z(\mathbf{x},t)}{\int d^d x' Z(\mathbf{x}',t)}. \tag{5}$$

For future convenience we denote by $P_0(\mathbf{x},t)$ the probability density of a directed line in the absence of any external potential (which is actually equal to the diffusion equation Green function). One may also define a local ''free energy'' via

$$f(\mathbf{x},t) = -T\,\ln[Z(\mathbf{x},t)]. \tag{6}$$

We use the term ''free energy'' guardedly, since this quantity, as defined above, is not extensive (in terms of the length t of the lines). It is, however, a useful measure of the energy-entropy balance for a given end-point value $\mathbf{x}$. For instance, if one chooses V to be a columnar potential (which exists only for $\mathbf{x}=\mathbf{0}$), then $f(\mathbf{0},t)$ is a sensitive measure of a bound line $(f\sim t)$, as opposed to an unbound line [generally $f\sim\ln(t)$]. In this paper we shall generally intuit the behavior of the line from studying the probability density.

As a final remark, we stress that although the path-integral and partial differential equation forms for Z bear a striking resemblance to the Feynman path-integral and Schrödinger equation descriptions of a quantum mechanical particle, respectively, one should use caution in applying results from quantum mechanics to the present problem — the idea that the problems are related by a simple Wick rotation $(\tau = it)$ is oversimplified. Physically, the partition function as defined above is a positive-definite quantity. There is no analogous constraint on the complex wave function of quantum mechanics. Also, the time evolution of the wave function is generally postulated to be continuous [16]. In the above

problem, there is no physical constraint on the continuity of the partition function as a function of t. The reason for this difference is the following. In quantum mechanics a discontinuity in the time evolution of the wave function corresponds to a temporal discontinuity in the probability density of the quantum mechanical particle, which is physically unreasonable. In contrast, the discontinuity of the partition function breaks no physical laws. This is because the symbol t used above corresponds to the *length of the line* — this is a fixed quantity for a given line. The meaning of the rate of change of the partition function with respect to t is to take two lines whose lengths are t and $t+\delta t$, respectively, and to equilibrate them in identical thermal (and possibly quenched disorder) environments; thereafter measuring the partition function for each line. There is no reason *a priori* to insist on continuity of the partition function.

These general distinctions between quantum mechanics (as expressed by Feynman's path integral) and the statistical mechanics of a directed line [as expressed by (1) above] will have important consequences in the remainder of this paper.

III. DISCONTINUITIES AND ''GENERAL SOLUTION''

To gain some insight into the nature of a sparse potential, let us simplify the problem to having a single potential $v(\mathbf{x})$ located at a longitudinal position τ. The integral equation for Z now takes the form

$$Z(\mathbf{x},t)=g(\mathbf{x},t)-\int d^dx'\int_0^t dt'\,g(\mathbf{x}-\mathbf{x}',t-t')v(\mathbf{x}')\times\delta(t'-\tau)Z(\mathbf{x}',t'). \tag{7}$$

Obviously, for $t<\tau$ we have the solution $Z(\mathbf{x},t)=g(\mathbf{x},t)$, which implies $P(\mathbf{x},t)=P_0(\mathbf{x},t)=g(\mathbf{x},t)$. In particular, $Z^-(\mathbf{x})\equiv\lim_{\epsilon\to0}Z(\mathbf{x},\tau-\epsilon)=g(\mathbf{x},\tau)$. We now go on to the more difficult task of finding $Z^+(\mathbf{x})\equiv\lim_{\epsilon\to0}Z(\mathbf{x},\tau+\epsilon)$.

In taking the limit $t\to\tau$ care is needed when integrating over the δ function in Eq. (7), but the Green function $g(\mathbf{x}-\mathbf{x}',t-t')=g(\mathbf{x}-\mathbf{x}',t-\tau)$ may safely be replaced by $\delta(\mathbf{x}-\mathbf{x}')$ leading to

$$Z^+(\mathbf{x})=g(\mathbf{x},\tau)-v(\mathbf{x})\lim_{\epsilon\to0}\int_0^{\tau+\epsilon}dt'\,\delta(t'-\tau)Z(\mathbf{x},t'). \tag{8}$$

Therefore $Z^+(\mathbf{x})=g(\mathbf{x},\tau)+O(v)$, and since this also holds for $Z^-(\mathbf{x})$ to this order, we may write Eq. (8) as

$$Z^+(\mathbf{x})=g(\mathbf{x},\tau)-v(\mathbf{x})\lim_{\epsilon\to0}\int_0^{\tau+\epsilon}dt'\,\delta(t'-\tau)g(\mathbf{x},\tau)+O(v^2)=g(\mathbf{x},\tau)[1-v(\mathbf{x})+O(v^2)]. \tag{9}$$

It is important to note that, while one can expand the function g about $t=\tau$, this is not so for Z, which can now clearly be seen to be discontinuous at this point.

To find the higher order terms in Eq. (9), we write $Z^+(\mathbf{x})$ and $Z^-(\mathbf{x})$ in the unified form:

$$Z(\mathbf{x},t)=g(\mathbf{x},\tau)-v(\mathbf{x})\int_0^t dt'\,\delta(t'-\tau)Z(\mathbf{x},t'), \tag{10}$$

where $t=\tau\pm\epsilon,\epsilon\to0$. Iterating this equation gives

$$Z(\mathbf{x},t)=g(\mathbf{x},\tau)\left\{1-\int_0^t dt'\,v\,\delta(t'-\tau)+\int_0^t dt'\int_0^{t'}dt''\times v^2\delta(t'-\tau)\,\delta(t''-\tau)+\cdots\right\}. \tag{11}$$

For $t<\tau$ the integrals give no contribution, confirming the result for $Z^-(\mathbf{x})$, and for $t>\tau$ the lower limits of the integrals may be taken to minus infinity and t replaced by plus infinity, giving

$$Z^+(\mathbf{x})=g(\mathbf{x},\tau)\left\{1-\int_{-\infty}^{\infty}dt'\,v\,\delta(t'-\tau)+\frac{1}{2}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}dt''\times v^2\delta(t'-\tau)\,\delta(t''-\tau)+\cdots\right\}, \tag{12}$$

where the symmetry of the double integral has been used to rewrite it as an integral over all t and t', hence the factor of 1/2. Continuing this procedure one finds that

$$Z^+(\mathbf{x})=g(\mathbf{x},\tau)\sum_{n=0}^{\infty}\frac{1}{n!}\left\{-\int_{-\infty}^{\infty}dt'\,v\,\delta(t'-\tau)\right\}^n. \tag{13}$$

Therefore, in the limit $\epsilon\to0$,

$$Z(\mathbf{x},t)=\begin{cases}g(\mathbf{x},\tau)\exp[-v(\mathbf{x})] & \text{if } t=\tau+\epsilon\\ g(\mathbf{x},\tau) & \text{if } t=\tau-\epsilon.\end{cases} \tag{14}$$

From Eq. (10) it is clear that, in the limit $t\to\tau$, the coordinate x is simply a label and plays no significant part in the phenomenon we have just highlighted. Therefore, some further insight into this effect may be gained by a study of the zero-dimensional versions of these models. In order to do this it is useful to examine the differential equations corresponding to the above integral equations. The one corresponding to Eq. (7) has the form [cf. Eq. (2)]

$$\partial_t Z(\mathbf{x},t)=\nu\nabla^2Z-v(\mathbf{x})\,\delta(t-\tau)Z. \tag{15}$$

The zero-dimensional version of this equation is

$$\partial_t Z(t)=-v\,\delta(t-\tau)Z. \tag{16}$$

It is easy to check that the integral equation corresponding to Eq. (16) is simply Eq. (10) but with the x label absent. On the other hand, we can in this simple case, solve Eq. (16) directly to find:

$$Z(t)=\begin{cases}g(\tau)\exp(-v) & \text{if } t=\tau+\epsilon\\ g(\tau)\exp[-v\,\theta(0)] & \text{if } t=\tau\\ g(\tau) & \text{if } t=\tau-\epsilon\end{cases} \tag{17}$$

in the limit $\epsilon\to0$. We conclude that, while one may meaningfully formulate questions about the discontinuity of Z at $t=\tau$, the definition of the partition function is itself ill defined precisely at this point, depending as it does on the definition of $\theta(0)\equiv\int_{-\infty}^0 dt\,\delta(t)$. Of course, in a microscopic approach, the δ function would be smoothed out and this

ambiguity would be absent. However, as far as the evolution of Z is concerned, only its change at $t=\tau$, and not its actual value there, is relevant. So the conclusions that we draw in this paper will be independent of the precise form of any underlying microscopic model.

Returning to the general form of the potential Eq. (3), the relation

$$Z^+(\mathbf{x},\tau)=Z^-(\mathbf{x},\tau)\exp[-v(\mathbf{x})] \tag{18}$$

allows us to write a general iterative solution for the partition function. The idea is to split the evolution of Z into two parts; the first being concerned with the change in Z as the line encounters a sparse potential, the second with the evolution of Z between potentials. We naturally define

$$Z_n^-(\mathbf{x})=\lim_{\epsilon\to 0} Z(\mathbf{x},\tau_n-\epsilon), \tag{19}$$

and also

$$Z_n^+(\mathbf{x})=\lim_{\epsilon\to 0} Z(\mathbf{x},\tau_n+\epsilon). \tag{20}$$

Directly using Eq. (18) we have

$$Z_n^+(\mathbf{x})=Z_n^-(\mathbf{x})\exp[-v_n(\mathbf{x})]. \tag{21}$$

The evolution of the partition function between potentials is easily obtained since it is nothing but thermal wandering. We therefore have

$$Z_{n+1}^-(\mathbf{x})=\int d^dx'\, g(\mathbf{x}-\mathbf{x}',\tau_{n+1}-\tau_n)Z_n^+(\mathbf{x}'). \tag{22}$$

Equations (21) and (22) are the main results of this section and constitute an iterative solution for the partition function, in some ways analogous to the usual transfer matrix solution used in discrete lattice formulations of directed walks [1]. Once the set of functions Z_n^+ and Z_n^- is known, the partition function at intermediate values of the longitudinal coordinate may be found by quadrature from Eq. (4).

IV. SINGLE DEFECT

In this section we solve perhaps the simplest example of a sparse potential, namely, a short-ranged potential corresponding to a single point defect, located at longitudinal location $s=\tau$. For convenience we choose $v(\mathbf{y})=-\rho\Delta(\mathbf{y})$, where

$$\Delta(\mathbf{y})=(\pi a^2)^{-d/2}\exp(-y^2/a^2). \tag{23}$$

The scale a is to be regarded as the shortest transverse scale in the problem, although we shall always need to keep a nonzero in order to regularize the theory. (Note, in the limit of $a\to 0$, the function Δ becomes a Dirac δ function.) The parameter ρ simply represents the strength of the ''defect'' — for $\rho>0$ the defect is attractive, while for $\rho<0$, the defect is repulsive. We can now go on to calculate the probability densities.

Since the only potential in the system is that due to the single defect, we clearly have $Z^-(\mathbf{x},\tau)=g(\mathbf{x},\tau)$, which implies $P^-(\mathbf{x},\tau)=P_0(\mathbf{x},\tau)$. Integrating $Z^+(\mathbf{x},\tau)$, as given in Eq. (14), over $\mathbf{x}$ in order to find the appropriate normalization leads us to an expression for the probability density of the line on the positive side of the defect. It is convenient when discussing this quantity to introduce a length $l=(4\nu\tau)^{1/2}\gg a$, which is the effective transverse wandering of the line between the origin and the defect, and then to form appropriately scaled versions of ρ and l by defining $\rho^*=\rho/(\pi a^2)^{d/2}$ and $l^*=l/a$. One then finds that

$$P^+(\mathbf{x},\tau)=P_0(\mathbf{x},\tau)\left\{\frac{(l^*)^d\exp\rho\Delta(x)}{e^{\rho^*}+(l^*)^d-1}\right\}. \tag{24}$$

Looking at the short-range form of this expression it is immediately clear that

$$\frac{P^+(\mathbf{0},\tau)}{P_0(\mathbf{0},\tau)}\sim\left\{\frac{(l^*)^d e^{\rho^*}}{e^{\rho^*}+(l^*)^d-1}\right\}, \tag{25}$$

while for $|\mathbf{x}|\to\infty$ we have

$$\frac{P^+(\mathbf{x},\tau)}{P_0(\mathbf{x},\tau)}\sim\left\{\frac{(l^*)^d}{e^{\rho^*}+(l^*)^d-1}\right\}. \tag{26}$$

Examination of these expressions for the relative discontinuity of the probability density reveals the following effects:

Attractive weak defect $\rho>0$ *and* $\rho=O(a^d)$: In this case $\rho^*=O(1)$, and since $l^*\gg 1$, the short-range discontinuity of the probability density [as given by Eq. (25)] is of order unity, while the long-range discontinuity [as given by Eq. (26)] is of negligible size.

Attractive strong defect $\rho>0$ *and* $\rho=O(1)$: The situation is markedly different here: for small $|\mathbf{x}|$, $P^+\gg P_0$ and for large $|\mathbf{x}|$, $P^+\ll P_0$. The two probability densities become equal at some critical value of $|\mathbf{x}|$ which is much smaller than a.

Repulsive defect $\rho<0$: The conclusions for weak repulsive defects are exactly as for weak attractive defects. For strong repulsive defects $P^+\ll P_0$ at short range and $P^+\approx P_0$ at long range.

These results can be summarized by saying that at long range (in practice for $|\mathbf{x}|>a$) weak attractive defects and all repulsive defects have no effect. But a strong attractive defect does have a global effect on the probability density of the directed line, at the longitudinal location τ.

We shall briefly consider the form of the probability density for $t>\tau$. To do this we write down an integral equation of the form Eq. (7), but with initial time τ_+. Using $Z^+(\mathbf{x},\tau)$, as given by Eq. (14), we find

$$\frac{P(\mathbf{x},t)}{P_0(\mathbf{x},t)}\approx\frac{(l^*)^d+F(\mathbf{x},t)(e^{\rho^*}-1)}{(l^*)^d+e^{\rho^*}-1}, \tag{27}$$

where

$$F(\mathbf{x},t)=\frac{1}{(1-\gamma)^{d/2}}\exp\left[-\frac{\mathbf{x}^2}{4\nu t^2}\frac{\tau}{(1-\gamma)}\right] \tag{28}$$

and where we have defined $\gamma(t)=\tau/t$.

Let us examine the consequences of this result for times significantly greater than τ; i.e., we take $\gamma \ll 1$. Firstly, we note that the effect of the defect upon the probability density is negligible, if the defect is repulsive or attractive and weak, since the quantity $(l^*)^d$ dominates in both the numerator and denominator of Eq. (27). The situation is again more interesting when we consider a strong attractive defect. It is now the quantity e^{ρ^*} that dominates, at least for small enough x. Therefore, the healing of the distribution function at $\mathbf{x}=\mathbf{0}$ follows $(P/P_0)_{x=0} \sim F(\mathbf{0},t) \sim 1 + d\tau/2t$. For transverse distance $|\mathbf{x}|$ being large [actually $|\mathbf{x}| \gg t(\nu/\tau)^{1/2}$], the healing does not occur except at extremely large longitudinal distances; the relative difference in the probability density (compared to the free case) satisfying $(P/P_0)_{x=\infty} \sim (l^*)^d e^{-\rho^*}$. There will exist a scale $L(t)$ at which the ratio of P to P_0 is exactly unity, given by $F(\mathbf{x},t) \sim 1$. From Eq. (28) we find $L(t) = (2d\nu t)^{1/2}$.

So, to summarize the results for the simple situation of a single localized defect of strength ρ, we find that there is a qualitative difference between attractive and repulsive defects. In the former case, there exist two classes of defect — weak and strong — which are distinguished by their effect upon the probability density, this effect being local and global, respectively. In the latter case (repulsive defect), we find that for any strength of defect, the effect upon the probability density is always local. The extreme asymmetry in effect of positive and negative localized defects will be seen to have interesting consequences in the next two sections in which we consider infinite arrays of defects. Since the effect of a repulsive defect upon the line is qualitatively the same for any strength of defect, we shall not distinguish between weak and strong repulsive defects. In the following sections we shall generally take the strength of the repulsive defect to be of order unity.

V. PERIODIC COLUMNAR POTENTIAL I

In this section we shall consider a more complicated situation, namely, an infinite periodic array of localized defects located on the column defined by $\mathbf{x}=0$. We henceforth restrict our attention to $(2+1)$ dimensions. This shares the attractive features of being both the most physically interesting case as well the most analytically tractable — a rare coincidence. Choosing τ to be the longitudinal separation between the defects, and ρ to be their strength, we consider a potential of the form

$$V(\mathbf{y},s) = -\rho \sum_{n=1}^{\infty} \Delta(\mathbf{y})\, \delta(s-\tau n), \tag{29}$$

where we adopt the Gaussian envelope form (23) for the short-range function Δ. The range of the potentials is of $O(a)$, which we take to be the smallest transverse scale in the problem. In particular we have $a \ll l$ where $l = (4\nu\tau)^{1/2}$ is the effective transverse wandering of the line between defects. All results will be derived to leading order in a; the fact that the transverse scale of the function $\Delta(\mathbf{x})$ is of $O(a)$ allows us to frequently implement it as a Dirac δ function to get results to this order. We have chosen all the defects to have the same strength ρ, which we can take to be either positive (attractive defects) or negative (repulsive defects). In the next section we shall study an analogous situation, but with alternating attractive and repulsive defects.

Taking the general iterative solution as given in Eqs. (21) and (22) and substituting the explicit form for the potential above yields the relations

$$Z^-_{n+1}(\mathbf{x}) = \int d^d x'\, g(\mathbf{x}-\mathbf{x}',\tau) Z^+_n(\mathbf{x}'), \tag{30}$$

and

$$Z^+_n(\mathbf{x}) = Z^-_n(\mathbf{x}) \exp[\rho\Delta(\mathbf{x})]. \tag{31}$$

Combining these two results and implementing Δ as a Dirac δ function wherever possible we find

$$\begin{aligned} Z^-_n(\mathbf{x}) &= \frac{Z^-_{n-1}(\mathbf{0})}{\Delta(\mathbf{0})} g(\mathbf{x},\tau)[e^{\rho\Delta(\mathbf{0})}-1] \\ &\quad + \int d^d x'\, g(\mathbf{x}-\mathbf{x}',\tau) Z^-_{n-1}(\mathbf{x}') \\ &= \frac{Z^-_{n-1}(\mathbf{0})}{\Delta(\mathbf{0})} g(\mathbf{x},\tau)[e^{\rho\Delta(\mathbf{0})}-1] + \int d^d x' \\ &\quad \times g(\mathbf{x}-\mathbf{x}',2\tau) Z^-_{n-2}(\mathbf{x}') \exp\rho\Delta(\mathbf{x}'). \end{aligned} \tag{32}$$

Repeating this procedure leads us to

$$Z^-_n(\mathbf{x}) = g(\mathbf{x},n\tau) + R \sum_{m=1}^{n-1} g[\mathbf{x},(n-m)\tau] Z^-_m(\mathbf{0}). \tag{33}$$

where R is defined by

$$R = \frac{e^{\rho\Delta(\mathbf{0})}-1}{\Delta(\mathbf{0})}. \tag{34}$$

It is convenient at this point to define $\psi_n = Z^-_n(\mathbf{0})$, along with $f_n = g(\mathbf{0},\tau n) = 1/(\pi n l^2)$. Setting $\mathbf{x}=\mathbf{0}$ in the above equation then gives

$$\psi_n = f_n + R \sum_{m=1}^{n-1} f_{n-m}\psi_m . \tag{35}$$

This discrete equation may be solved exactly by making use of a generating function. The details of the calculation are relegated to Appendix A. The resulting form for ψ_n depends on whether defects are attractive or repulsive. Thus we consider these two cases separately.

A. Attractive defects

From Appendix A, the asymptotic (i.e., $n \gg 1$) result for ψ_n takes the form

$$R\psi_n \sim \left[\frac{e^{-1/R'}}{R'(1-e^{-1/R'})}\right] \exp\left[n \ln\left(\frac{1}{1-e^{-1/R'}}\right)\right], \tag{36}$$

where $R' = R/\pi l^2 = [e^{\rho^*}-1]/(l^*)^2$. So, for any $\rho > 0$, the partition function, evaluated at a defect site, grows exponentially; but with a rate that vanishes exponentially fast for

small ρ. In order to obtain the physically meaningful probability density, we must normalize the partition function. Generally, the normalization is defined as $N(t) = \int d^d x Z(\mathbf{x},t)$. In particular, we define

$$N_n = \lim_{\epsilon \to 0} N(\tau n - \epsilon). \tag{37}$$

Then by integrating Eq. (33) over the transverse space we obtain

$$N_n = 1 + R \sum_{m=1}^{n-1} \psi_m, \tag{38}$$

which on evaluating the sum gives, for large n,

$$N_n \sim (1/R') \exp\left[n \ln\left(\frac{1}{1 - e^{-1/R'}} \right) \right]. \tag{39}$$

Dividing the partition function Eq. (36) by this normalization (39) yields the asymptotic form for the probability density at the nth defect:

$$P_n(\mathbf{0}) \equiv \lim_{\epsilon \to 0} P(\mathbf{0}, \tau n - \epsilon) \sim \frac{e^{-1/R'}}{R(1 - e^{-1/R'})}. \tag{40}$$

From this result we see that for any $\rho > 0$, the probability density on the column (actually on a defect site) attains a nonzero, constant value as $n \to \infty$. This indicates that the line is always *bound* to the array of defects, regardless of how weakly attractive they are, or how widely separated.

It is interesting to calculate the probability density on the column, but in between the defects. This may be directly evaluated by making use of Eq. (4). Setting $t = \tau(n + \theta)$ with $\theta \in (0,1]$ we find the asymptotic result

$$P(\mathbf{0},t) \sim \frac{R' e^{-1/R'}}{R(1 - e^{-1/R'})^{\theta}} F_{\theta}(-\ln(1 - e^{-1/R'})), \tag{41}$$

where

$$F_{\theta}(p) \equiv \int_p^{\infty} du \frac{e^{-u\theta}}{(1 - e^{-u})}. \tag{42}$$

On setting $\theta = 1$, the above expression reduces to the asymptotic result for defect sites, as given by Eq. (40).

In the limit of very strong attractive defects, the probability density along the column has the form $P(\mathbf{0},(n+\theta)\tau) = (\theta \pi l^2)^{-1}$ for $0 < \theta \leqslant 1$. The density takes its largest value on the positive side of the defect, and then decays as $1/\theta$ until the next defect is reached ($\theta = 1$).

We also note that in the limit of vanishing defect strength $P(\mathbf{0},t)$ reduces to the same form for both defect sites, and positions in between. Explicitly one has

$$P(\mathbf{0},t) \sim e^{-(l^*)^2/\rho^*}/\rho, \quad \rho^* \ll 1, \quad t \gg \tau. \tag{43}$$

In the sense that the array binds the line for any $\rho > 0$, we may say that it acts in precisely the same way as a constant energy column (CEC), which is attractive [8,11,12]. In that case, one has a potential of the form $V(\mathbf{y},s) = -\overline{\rho}\overline{\Delta}(\mathbf{y})$. The asymptotic form of the probability density on the column, for vanishingly small $\overline{\rho}$, takes the form

$$P(\mathbf{0},t) \sim e^{-1/\widetilde{\rho}}/a^2 \widetilde{\rho}, \tag{44}$$

where $\widetilde{\rho} = \overline{\rho}/(4\pi\nu)$. Comparing these results, we see that concerning the dominant exponential behavior, there is an effective renormalization of the defect strength, such that it appears as the strength of a CEC. The precise form of this renormalization is $\overline{\rho} = \rho/\tau$, which is an intuitively appealing result.

B. Repulsive defects

The solution to this discrete equation for the partition function is outlined in Appendix A, along with details of the evaluation of the normalization. On dividing ψ_n by N_n, we obtain the probability density at a defect site. We find it to have the asymptotic form

$$P_n(\mathbf{0}) = \{ n \pi l^2 (R')^2 [\ln(n e^{1/|R'|})]^2 \}^{-1} = \begin{cases} [n \pi l^2]^{-1}, & 1 \ll n \ll e^{1/|R'|} \\ [\pi l^2 (R')^2 n [\ln(n)]^2]^{-1}, & n \gg e^{1/|R'|}. \end{cases} \tag{45}$$

So in the deep asymptotic regime, the probability density at a defect location decays as $P_n(\mathbf{0}) \sim \{ n [\ln(n)]^2 \}^{-1}$. Again, it is interesting to compare this result to that obtained for the case of a CEC (this time with repulsive energy). Following the methods of Ref. [12] one may ascertain that for the CEC, $P(\mathbf{0},t) \sim [t \ln(t)]^{-1}$ in the asymptotic regime. It therefore appears as if the defects repel the line more effectively than a CEC; which is counterintuitive. The situation may be clarified by calculating the probability density in between the defects, i.e., taking $t = \tau(n + \theta)$ with $n \gg 1$. This may be done by making use of Eq. (4) with the result that

$$P(\mathbf{0},t) \sim \frac{1}{t} \left[1 + O\left(\frac{1}{\ln(t)} \right) + O\left(\frac{1}{\theta [\ln(t)]^2} \right) \right]. \tag{46}$$

This is the asymptotic behavior of a free line. Therefore the repulsive array has no qualitative effect upon the probability density except right at the defect positions. The decay of $P(\mathbf{0},t)$ for the CEC is marginally faster than a free line, but marginally slower than that of a line constrained to pass through a defect — this is an intuitively acceptable result. In contradistinction to the case of attractive defects, there is no effective renormalization of the defects into a CEC when they are repulsive.

VI. PERIODIC COLUMNAR POTENTIAL II

In the last section we have seen that there exists a great difference between an infinite array of attractive defects, and an infinite array of repulsive defects. In the former case, the line is bound to the array; and for small values of the potential energy, the array acts precisely in the same manner as a CEC. In the latter case, the line is oblivious to the column on which the array is defined, except directly at defect sites. In that case, the probability density is marginally reduced. There is no relation between the repulsive array, and a repulsive CEC. One may ask how the line acts when the array

consists of both attractive and repulsive defects. In fact our initial motivation was to study the case of a random admixture of such defects along the column. However, this simple example of a disordered potential is extremely difficult to analyze. A simpler task is to arrange the attractive and repulsive defects in an alternating pattern along the column. The physics of such a system has attracted much interest recently [13,14], albeit in a microscopic formulation in terms of the RSOS model. Our presentation in this section will be rather brief as most of the calculation may be constructed using the methods of the past two sections. Also, we shall content ourselves with only examining the gross features of this system; namely, the location of the binding-unbinding transition, and the qualitative behavior of the critical properties of the line and the bound phase.

Denoting the strength of the attractive defects by $\rho>0$, and that of the repulsive defects by $-\sigma<0$, we consider the set of sparse potentials

$$V(\mathbf{y},s)=-\rho\sum_{n=1}^{\infty}\Delta(\mathbf{y})\,\delta(s-2n\tau)+\sigma\sum_{n=1}^{\infty}\Delta(\mathbf{y})\,\delta(s-(2n-1)\tau). \tag{47}$$

Following a similar procedure to that used in the previous section we may derive a closed equation for the partition function at a defect site. In this case there is the minor complication of having two types of defects, which may be easily accommodated in the following way. We denote the partition function at attractive and repulsive defect sites by ψ_n^E and ψ_n^O, respectively: $\psi_n^E\equiv Z_{2n}^{-}(\mathbf{0})$ and $\psi_n^O\equiv Z_{2n-1}^{-}(\mathbf{0})$ [the indices E and O represent ''even'' and ''odd'' (in terms of sites $2n$ and $2n-1$)]. To derive the iterative equation for these quantities, one uses the fundamental relations (21) and (22) and follows a similar procedure to that described in the previous section. This leads to the following simultaneous equations:

$$\psi_n^E=f_{2n}+R\sum_{m=1}^{n-1}f_{2n-2m}\psi_m^E-S\sum_{m=1}^{n}f_{2n-2m+1}\psi_m^O,$$

$$\psi_n^O=f_{2n-1}+R\sum_{m=1}^{n-1}f_{2n-2m-1}\psi_m^E-S\sum_{m=1}^{n-1}f_{2n-2m}\psi_m^O, \tag{48}$$

where R is given by Eq. (34) and

$$S=\frac{1-e^{-\sigma\Delta(\mathbf{0})}}{\Delta(\mathbf{0})}. \tag{49}$$

We shall relegate the explicit solution of these equations to Appendix B. The main result to emerge from this solution is the shifting of the binding-unbinding transition to a critical line in the (ρ,σ) plane which is illustrated in Fig. 1. The precise equation for this line is given in Eq. (B12), but the general structure takes the form

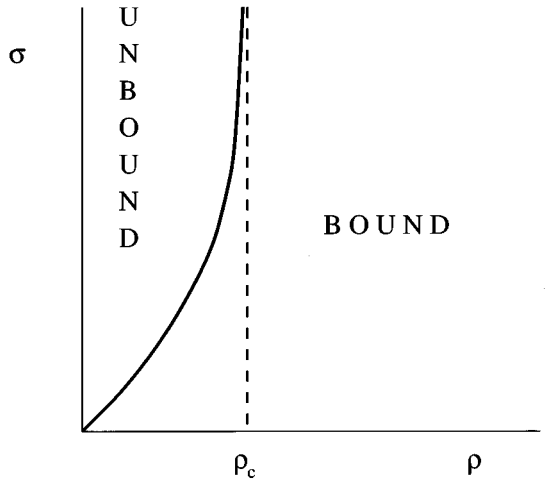

FIG. 1. The phase diagram in the (σ,ρ) plane. The critical line separating unbound and bound phases is given by Eq. (50).

$$\sigma=\begin{cases}\rho, & \text{if } \rho\ll\rho_c\\ -\pi a^2\ln\left(\dfrac{\rho_c-\rho}{\pi a^2}\right), & \text{if } \rho\to\rho_c,\end{cases} \tag{50}$$

where $\rho_c=\pi a^2\ln 2$. It is interesting to note that for $\rho>\rho_c$ the line is always bound, regardless of the strength of the repulsive sites. This result is in qualitative agreement with the recent work from the Fribourg group [13,14]. This physics may have been intuitively expected following the analysis of the single defect, where the vast difference of effect between attractive and repulsive defects was examined in detail. In the terminology of Sec. IV, we may say that strong attractive defects will always bind the line, whereas weak attractive defects require a critical strength in order to do this. There is no analogy of ''weak'' and ''strong'' for repulsive defects.

Some further details are examined in Appendix B, which we shall summarize here. Firstly one may examine the behavior of the line at criticality. In this case one finds that the probability density of the line at a defect site (attractive or repulsive) follows the asymptotic behavior of a free line, namely, $P(\mathbf{0},n\tau)\sim 1/n$. One may also study the bound state in which case one finds that the probability density saturates to a constant at both attractive and repulsive defect sites, although the density is a factor of $(1/l^*)^2$ smaller at the repulsive defect sites. One then has the picture that, although the line is bound, it really binds only to the attractive defects, and has a vanishingly small probability density at the repulsive defect sites. Having gleaned the main qualitative aspects of the alternating column, we shall end this section here and proceed to presenting our conclusions.

VII. CONCLUSIONS

In this paper we have examined a class of models described as ''a directed line in the presence of sparse potentials,'' with the understanding that a sparse potential is a d-dimensional potential defined at a single longitudinal location. In Sec. III we obtained a general iterative solution for the partition function that consisted of two pieces: free propagation between potentials, and a discontinuity when

passing through a potential. In Sec. IV we considered in some detail a single short-ranged sparse potential that corresponds to a single defect. The probability density of the line was evaluated for both attractive and repulsive defects. In the former case, the density was seen to undergo a global discontinuous change for strong defects, and the healing of the density (i.e., the relaxation to the density of a free line) was found to be incomplete for arbitrarily large longitudinal distances above the defect. The latter case of a repulsive defect was completely different in that for any strength of defect, the density of the line is undisturbed except within a small region about the defect. [As a brief aside we may relate this extreme asymmetry to the behavior of a nonequilibrium interface evolving under the Kardar-Parisi-Zhang (KPZ) equation [6]. Under the mapping between directed lines and the KPZ equation, an attractive (repulsive) defect corresponds to an upward (downward) force, acting for a short duration upon the surface. It has been previously observed [12,18] that in the strong-coupling regime of the KPZ equation, an upwards force of sufficient strength may seed a large disturbance in the interface, which then becomes effectively frozen. Alternatively, a downward force of arbitrarily large strength plays no role, since any disturbance it causes is quickly eradicated by the strong upward action of the KPZ nonlinearity. The behavior of the directed line under the influence of a single defect is seen to exhibit an analogous effect. The possibility of gaining intuition concerning the strong-coupling behavior of the KPZ equation is a prime example of the usefulness of studying nontrivial, yet tractable, directed line models of the type considered in this paper.]

In Sec. V we studied a periodic array of defects arranged on a column, exclusively in $(2+1)$ dimensions, which is of most interest. For attractive defects, the line was found to be asymptotically bound, and indeed, the saturated form of the probability density along the column (for vanishingly weak potentials) was found to correspond to that obtained previously for a CEC, indicating that the line samples the defects in such a way as to renormalize their effect to be that of a CEC. For an array consisting of repulsive defects, we found that the density along the column is qualitatively unchanged from that of a free line, i.e., $P \sim 1/t$. This result is logarithmically modified at the defect positions, having the form $P \sim 1/[t \ln^2(t)]$. There is no relation of these results to a repulsive CEC where one has $P \sim 1/[t \ln(t)]$, indicating that the sampling of the defects by the line does not have a renormalizing, or smoothing effect. (The fact that the line may be bound by an array of attractive defects has a novel implication for the KPZ equation; namely, that a sequence of discrete upward impulses is sufficient to move the interface with nonzero velocity, similar to the effect of pushing with a constant force [12]). In the last section we examined an array consisting of alternating attractive (with strength $\rho > 0$) and repulsive (strength $-\sigma < 0$) defects. It was found that the binding-unbinding transition is shifted to a location in the (σ, ρ) phase plane defined by the condition given in Eq. (50). This result is interesting as it indicates that for attractive defects of strength greater than the critical strength $\rho_c = \pi a^2 \ln(2)$, the line will always be bound, regardless of the strength of the repulsive defects. This latter result is in accord with recent calculations on an equivalent microscopic RSOS model [13,14]. The asymptotic behavior of the line at criticality was found to be simply that of a free line $[P(\mathbf{0}, n\tau) \sim 1/n]$. In the bound phase the line was found to be essentially bound to the attractive defects, the density at the repulsive defects being smaller by a factor of $(1/l^*)^2$.

We feel that the introduction of sparse potentials introduces some simplifying features into the study of directed lines. In the current paper we have examined probably the simplest form for these potentials, namely, single defects, and periodic arrays of defects; although even for these simple periodic arrays there are many more features that may be examined, such as the spatial variation of the probability density away from the column, and the behavior of the system in dimensions other than $(2+1)$. One may also retain the simplifying nature of a periodic array of sparse potentials, but relax the condition used in this paper that the potentials are short ranged in the transverse directions. For instance, it would be of interest to study potentials that were periodic in the transverse dimensions, as one may then make contact to systems in which a directed line is interacting with a set of crystal layer potentials. Part of our motivation for examining sparse potentials was to see if analytic progress is possible for simple types of disorder — such as randomly placed defects, or regularly placed defects with random energy. We certainly regard such analyses as worthwhile and possible projects for the future.

ACKNOWLEDGMENTS

We are grateful to Alan Bray for useful comments. T.J.N. acknowledges financial support from the Engineering and Physical Science Research Council.

APPENDIX A

In this Appendix we outline the solution of the recurrence relation for the partition function evaluated at the defect sites, ψ_n. This is given by Eq. (35):

$$\psi_n = f_n + R \sum_{m=1}^{n-1} f_{n-m} \psi_m . \tag{A1}$$

The solution is most easily obtained by introducing the generating function

$$\overline{\psi}(z) \equiv \sum_{n=1}^{\infty} z^n \psi_n , \tag{A2}$$

along with a similar function $\overline{f}(z)$ defined in terms of $\{f_n\}$. Summing (A1) over n with the appropriate weight then gives

$$\overline{\psi}(z) = \frac{\overline{f}(z)}{[1 - R\overline{f}(z)]} . \tag{A3}$$

Inverting the relation (A2) using the calculus of residues then yields the solution

$$\psi_n = \frac{1}{2\pi i} \oint_{C_1} \frac{dz}{z^{n+1}} \overline{\psi}(z) = \frac{1}{2\pi i R} \oint_{C_1} \frac{dz}{z^{n+1}} \frac{1}{[1 - R\overline{f}(z)]} , \tag{A4}$$

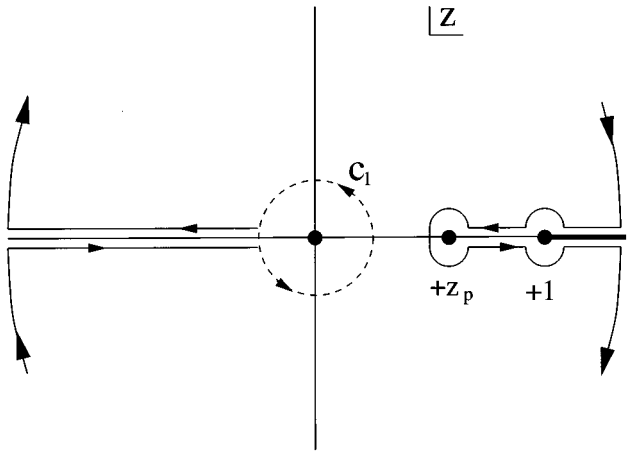

FIG. 2. The singularity structure in the complex z plane for the evaluation of $\psi(z)$. Also illustrated is the deformation (solid line) of the original contour C_1 (dashed line.)

where the contour C_1 is a closed circle of radius δ; this circle is chosen so that no other singularities are enclosed bar the nth order pole at the origin.

The function $f(z)$ is easily evaluated to be

$$\overline{f(z)}=(\pi l^2)^{-1}\sum_{n=1}^{\infty}\frac{z^n}{n}=-(\pi l^2)^{-1}\ln(1-z), \quad \text{(A5)}$$

where the sum is guaranteed to converge since $|z|<\delta\ll 1$. It is the ease with which this sum may be evaluated in $(2+1)$ dimensions that makes this case the most analytically tractable. Defining $R'=R/(\pi l^2)$, we have the explicit form for ψ_n as

$$\psi_n=\frac{1}{2\pi i R}\oint_{C_1}\frac{dz}{z^{n+1}}\frac{1}{[1+R'\ln(1-z)]}. \quad \text{(A6)}$$

Examination of the integrand reveals that there exist two singularities in the complex plane apart from the pole at the origin. These are a branch point at $z=1$ along with a simple pole at

$$z_p=1-\exp(-1/R')=1-\exp\left\{-\frac{(l^*)^2}{e^{\rho^*}-1}\right\}. \quad \text{(A7)}$$

By cutting the contour C_1 on the negative real axis, we may wrap it around the rest of the complex plane as illustrated in Fig 2. In this way, we have

$$\oint_{C_1}+\int_{\text{cut}}+(\text{residue at } z_p)=0.$$

We have thus replaced the essentially perturbative expression (A6) by an expression that enables us to extract the strong-coupling behavior, if it exists (this actually depends on the existence of a pole at radius $\delta<|z|<1$).

Whether the residue from the pole at z_p dominates over the contribution from the branch cut depends on the value of $|z_p|$. It is clear from the form of the integrand that the integral will be dominated (for large n) by the pole if it lies within the unit circle. From Eq. (A7) it follows that $z_p<1-\exp(l^*)^2$ if the defects are repulsive and $0<z_p<1$ if the defects are attractive. Thus for attractive defects the pole dominates and a calculation of the residue leads directly to Eq. (36).

In the case of repulsive defects, the pole is seen to lie on the negative real axis far outside the circle $|z|=1$. Thus the residue at z_p yields only an exponentially decaying contribution and is subdominant to the contribution from the cut. We therefore have $\oint_{C_1}\sim-\int_{\text{cut}}$, which has the explicit form

$$\psi_n\sim\frac{1}{\pi l^2}\int_0^{\infty}\frac{dx}{(1+x)^{n+1}}\left\{\frac{1}{(\pi R')^2+(1+R'\ln x)^2}\right\}. \quad \text{(A8)}$$

Referring to Eq. (38) we may express the normalization in integral form. Explicitly one finds

$$N_n\sim 1+R'\int_0^{\infty}\frac{dx}{x}\left\{\frac{(1+x)^n-1}{(1+x)^{n+1}}\right\}\left\{\frac{1}{(\pi R')^2+(1+R'\ln x)^2}\right\}. \quad \text{(A9)}$$

We shall briefly describe the asymptotic evaluation of Eq. (A8). The integral form of the normalization (as well as the similar integrals that appear in the evaluation of $P(\mathbf{0},t)$ for $t\neq n\tau$) may be done in an analogous fashion. So referring to Eq. (A8), as a first step we scale x by n, and use the relation $\exp(p)=\lim_{n\to\infty}(1+p/n)^n$. We then have

$$\psi_n\sim\frac{1}{n\pi l^2(R')^2}\int_0^{\infty}dx\ e^{-x}\left\{\frac{1}{\pi^2+[\ln(x/\beta)]^2}\right\}, \quad \text{(A10)}$$

where $\beta=ne^{-1/R'}>ne^{(l^*)^2}\gg 1$. We now split the integration range into three regions and estimate the order of magnitude of the integral in each region. Region (i) is defined by $0<x<1/\beta$, and retaining only dominant terms for small x, we find (up to prefactors) $\psi_n^{(i)}\sim[\beta\ln^2(\beta)]^{-1}$. Region (ii) is defined by $1/\beta<x<\beta$. In this region we may drop $\ln(x)$ in comparison to $\ln(\beta)$, which gives $\psi_n^{(ii)}\sim[\ln^2(\beta)]^{-1}$. Region (iii) is defined by $x>\beta$ in which case $\psi_n^{iii}\sim e^{-\beta}$. So clearly the contribution from region (ii) dominates for large n. In a similar way, one may establish that the normalization has the asymptotic form of $N_n\sim O(1)+O[1/\ln(n)]$. Putting these results together gives the form of the probability density shown in Eq. (45).

APPENDIX B

In this Appendix we outline the solution of the simultaneous iterative equations (48). As before, it is convenient to use generating functions. Thus we define the functions

$$\overline{\psi^E}(z)\equiv\sum_{n=1}^{\infty}z^{2n}\psi_n^E, \quad \overline{\psi^O}(z)\equiv\sum_{n=1}^{\infty}z^{2n-1}\psi_n^O. \quad \text{(B1)}$$

We also define

$$\overline{f^E}(z)\equiv\sum_{n=1}^{\infty}z^{2n}f_{2n}, \quad \overline{f^O}(z)\equiv\sum_{n=1}^{\infty}z^{2n-1}f_{2n-1}. \quad \text{(B2)}$$

Summing the iterative equations over the appropriate weight and using the above definitions yields the algebraic equations

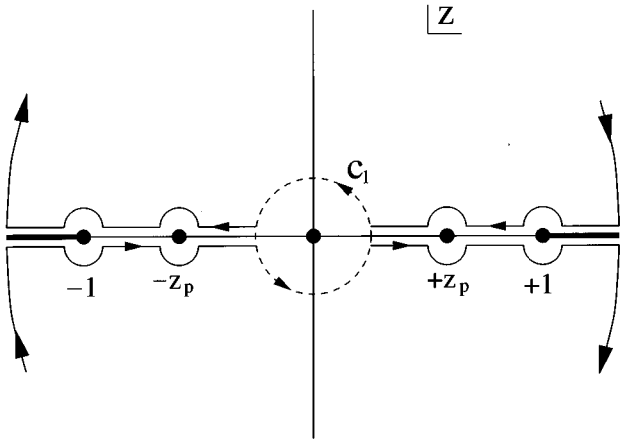

FIG. 3. The singularity structure in the complex z plane for the evaluation of $\overline{\psi^E}(z)$ and $\overline{\psi^O}(z)$. Also illustrated is the deformation (solid line) of the original contour C_1 (dashed line.)

$$\overline{\psi^E}(z)=\overline{f^E}(z)+R\overline{f^E}(z)\,\overline{\psi^E}(z)-S\overline{f^O}(z)\,\overline{\psi^O}(z), \quad \text{(B3)}$$

$$\overline{\psi^O}(z)=\overline{f^O}(z)+R\overline{f^O}(z)\,\overline{\psi^E}(z)-S\overline{f^E}(z)\,\overline{\psi^O}(z), \quad \text{(B4)}$$

which may be readily solved, yielding the solutions

$$\overline{\psi^E}(z)=\frac{\overline{f^E}(z)+S[\overline{f^E}(z)^2-\overline{f^O}(z)^2]}{1-(R-S)\overline{f^E}(z)-RS[\overline{f^E}(z)^2-\overline{f^O}(z)^2]}, \quad \text{(B5)}$$

and

$$\overline{\psi^O}(z)=\frac{\overline{f^O}(z)}{1-(R-S)\overline{f^E}(z)-RS[\overline{f^E}(z)^2-\overline{f^O}(z)^2]}. \quad \text{(B6)}$$

From the particular form of f_n, we also have

$$\overline{f^E}(z)=-\frac{1}{2\pi l^2}\ln(1-z^2), \quad \text{(B7)}$$

and

$$\overline{f^O}(z)=-\frac{1}{2\pi l^2}\ln\left(\frac{1-z}{1+z}\right). \quad \text{(B8)}$$

Given the definition of the generating functions, we may retrieve the original partition functions using

$$\psi_n^E=\frac{1}{2\pi i}\oint_{C_1}\frac{dz}{z^{2n+1}}\overline{\psi^E}(z), \quad \psi_n^O=\frac{1}{2\pi i}\oint_{C_1}\frac{dz}{z^{2n}}\overline{\psi^O}(z), \quad \text{(B9)}$$

where as before C_1 is a circle enclosing the origin of small enough radius such that it encloses no singularities other than the pole at the origin.

We now examine the singularity structure in the complex z plane. Firstly we note that there now exist two branch points at $z=\pm 1$, which we connect to infinity with cuts along the real axis as shown in Fig. 3. Also, any pole that may exist within the unit circle will have a twin reflected through the origin due to fact that the denominators of the generating functions are even functions of z. As before we replace the perturbative expression by the nonperturbative one by cutting the contour C_1 and extending around the singularities in the complex plane. This is illustrated in Fig. 3 and leads us to the expression

$$\oint_{C_1}+\int_{-\text{cut}}+\int_{+\text{cut}}+\sum_{z=\pm z_p}(\text{residues})=0.$$

The existence of a bound state will arise only from there being a pole within the unit circle. We thus examine the zeroes of the denominator of the generating functions — i.e., the positions of the poles are solutions of

$$1+\frac{1}{2}(R'-S')\ln(1-z^2)-R'S'\ln(1+z)\ln(1-z)=0, \quad \text{(B10)}$$

where we have defined $R'=R/(\pi l^2)$, as before, and similarly $S'=S/(\pi l^2)$. We analyze this equation by first noting that if ρ grows with a more slowly than a^2, in particular if $\rho=O(1)$, then there is no solution for any σ. On the other hand, if ρ grows like a^2, or faster, then so do R and S and the final term in Eq. (B10) plays only a subdominant role. Therefore the poles are situated at $\pm z_p$, with

$$z_p=\left[1-\exp\left(-\frac{2}{(R'-S')}\right)\right]^{1/2}. \quad \text{(B11)}$$

As we vary the parameters ρ and σ, the poles exist within the unit circle only when $R'>S'$. Therefore there is a critical line in the (ρ,σ) plane that separates the region where poles exist from the region where they do not, which has the equation $R'=S'$. A more precise equation for this line may be obtained by including the final term in Eq. (B10) in the analysis by substituting $z_p=1-\epsilon$ into the equation and solving it in the limit $\epsilon\to 0$. The resulting condition for criticality is now

$$S'=\frac{R'}{1+2\ln 2R'}. \quad \text{(B12)}$$

The right-hand side is a monotonically increasing function of R', which reaches the value $(l^*)^{-2}$ (corresponding to $\sigma\to\infty$) when $\rho=\pi a^2\ln 2\equiv\rho_c$. The general features of the critical line are now easy to find: a linear regime near the origin and a logarithmic approach to ρ_c from below. This behavior is summarized in Eq. (50) and illustrated in Fig. 1.

In order to examine the behavior of the line at criticality, we insert the critical condition (B12) into the expressions for the generating functions. One then finds that to leading order $\overline{\psi^E}(z)=\overline{f^E}(z)$ and $\overline{\psi^O}(z)=\overline{f^O}(z)$, which directly gives the partition function at the defect sites with no integration required; i.e., we have

$$\psi_n^E=1/(2n\pi l^2), \quad \psi_n^O=1/[(2n-1)\pi l^2]. \quad \text{(B13)}$$

The appropriate normalization factors may be derived in an analogous way to that described in Sec. V. Explicitly one defines

$$N_n^E=\lim_{\epsilon\to 0}\int d^dx\,Z(\mathbf{x},2n\tau-\epsilon), \quad \text{(B14)}$$

and

$$N_n^O = \lim_{\epsilon \to 0} \int d^d x \, Z[\mathbf{x}, (2n-1)\tau - \epsilon]. \qquad \text{(B15)}$$

These functions then satisfy the relations $N_n^E = N_n^O$, and

$$N_n^E = 1 + R \sum_{m=1}^{n-1} \psi_m^E - S \sum_{m=1}^{n} \psi_m^O. \qquad \text{(B16)}$$

At criticality these functions are asymptotically constants that along with the results for the partition functions (B13) lead to the asymptotic form of the probability density following $P(\mathbf{0}, n\tau) \sim 1/n$.

In order to examine the bound state one may simply ignore the subdominant contributions from the cuts, and evaluate the residues of the poles at $\pm z_p$. No explicit details of this calculation are given here as it may easily be reconstructed from the analogous case examined in Appendix A.

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